Example 9.7

A slender vertical bar of mass m is rigidly attached to a horizontal disk rotating with constant angular velocity ω_α (Fig. 9.23). What force and couple are exerted on the bar by the disk?

STRATEGY

The external forces and couples on the bar are its weight and the force and couple exerted on it by the disk. The angular velocity and acceleration of the bar are given and we can determine the acceleration of its center of mass, so we can use the Euler equations to determine the total force and couple.

SOLUTION

Choose a Coordinate System In Fig. (a) we place the origin of a body-fixed coordinate system at the center of mass with the y axis vertical and the x axis in the radial direction. With this orientation we will obtain simple expressions for the bar's angular velocity and the acceleration of its center of mass.

Draw the Free-Body Diagram We draw the free-body diagram of the bar in Fig. (a), showing the force **F** and couple **C** exerted by the disk.

Apply the Equations of Motion The acceleration of the center of mass of the bar due to its motion along its circular path is $\boldsymbol{a} = -\omega_0^2 b\mathbf{i}$. From Newton's second law,

$$\sum \mathbf{F} = \mathbf{F} - mg\mathbf{j} = m(-\omega_0^2 b\mathbf{i})$$

we obtain the force exerted on the bar by the disk :

$$\mathbf{F} = -m\omega_0^2 b\mathbf{i} + mg\mathbf{j}$$

The total moment about the center of mass is the sum of the couple **C** and the moment due to **F** :

$$\begin{aligned}\sum \mathbf{M} &= \mathbf{C} + \left(-\frac{1}{2}l\mathbf{j}\right) \times (-m\omega_0^2 b\mathbf{i} + mg\mathbf{j}) \\ &= C_x\mathbf{i} + C_y\mathbf{j} + \left(C_z - \frac{1}{2}mlb\omega_0^2\right)\mathbf{k}\end{aligned}$$

The bar's inertia matrix in terms of the coordinate system in Fig. (a) is

$$[I] = \begin{bmatrix} \frac{1}{12}ml^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{12}ml^2 \end{bmatrix}$$

and its angular velocity, $\omega = \omega_0\mathbf{j}$, is constant. The equation of angular motion, Eq. (9.29), is

$$\begin{bmatrix} C_x \\ C_y \\ C_z - \frac{1}{2}mlb\omega_0^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \omega_0 \\ 0 & 0 & 0 \\ -\omega_0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{12}ml^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{12}ml^2 \end{bmatrix} \begin{bmatrix} 0 \\ \omega_0 \\ 0 \end{bmatrix}$$

The right side of this equation equals zero, so the components of the couple exerted on the bar by the disk are $C_x=0$, $C_y=0$, $C_z=\frac{1}{2}mlb\omega_0^2$ . .

Alternative Solution The bar rotates about a fixed axis, so we can also determine the couple **C** by using Eq. (9.25). Let the fixed point ***O*** be the center of the disk (Fig. b), and let the body-fixed coordinate system be oriented with the *x* axis through the bottom of the bar. The total moment about ***O*** is

$$\sum \mathrm{M}_0=\mathbf{C}+(b\mathbf{i})\times(-m\omega_0^2 b\mathbf{i}+mg\mathbf{j})+(b\mathbf{i}+\frac{1}{2}l\mathbf{j})\times(-mg\mathbf{j})$$
$$=\mathbf{C}$$

Thus the only moment about ***O*** is the couple exerted by the disk. Applying the parallel-axis theorems, the bar's moments and products of inertia are (Fig. c)

$$I_{xx}=I_{x'x'}+(d_y^2+d_z^2)\,m=\frac{1}{12}ml^2+\left(\frac{1}{2}l\right)^2 m=\frac{1}{3}ml^2$$

$$I_{yy}=I_{y'y'}+(d_x^2+d_z^2)\,m=mb^2$$

$$I_{zz}=I_{z'z'}+(d_x^2+d_y^2)\,m=\frac{1}{12}ml^2+\left[b^2+\left(\frac{1}{2}l\right)^2\right]m=\frac{1}{3}ml^2+mb^2$$

$$I_{xy}=I_{x'y'}+d_x d_y m=0+b\left(\frac{1}{2}l\right)m=\frac{1}{2}mbl$$

$$I_{yz}=I_{y'z'}+d_y d_z m=0$$

$$I_{zx}=I_{z'x'}+d_z d_x m=0$$

,Substituting these results into Eq. (9.25), we obtain
,

$$\begin{bmatrix} C_x \\ C_y \\ C_z \end{bmatrix}=\begin{bmatrix} 0 & 0 & \omega_0 \\ 0 & 0 & 0 \\ -\omega_0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{3}ml^2 & -\frac{1}{2}mbl & 0 \\ -\frac{1}{2}mbl & mb^2 & 0 \\ 0 & 0 & \frac{1}{3}ml^2+mb^2 \end{bmatrix}\begin{bmatrix} 0 \\ \omega_0 \\ 0 \end{bmatrix}$$
$$=\begin{bmatrix} 0 \\ 0 \\ \frac{1}{2}mlb\omega_0^2 \end{bmatrix}$$